UDC 62-50

ON ENCOUNTER-EVASION GAME PROBLEMS

PMM Vol. 38, №2, 1974, pp. 211-223
A. G. CHENTSOV
(Sverdlovsk)
(Received August 6, 1973)

We consider a nonlinear encounter-evasion differential game on a finite time interval. To solve it we use an auxiliary program construction. The article is closely related to the investigations in [1 – 8].

1. Let the motion of a conflict-controlled system be described by equations of the form

$$dx/dt = f(t, x, u, v), \quad x[t_0] = x_0$$
$$x \in R^n, \quad u \in P \subset R^p, \quad v \in Q \subset R^q$$

The sets P and Q are assumed to be compact, while the function $f(\cdot)$ is assumed to be continuous in the aggregate and continuously differentiable in x. We assume that every solution $x(t)$ of the equation

$$dx(t)/dt \in \overline{\mathrm{co}}\, \{y: y = f(t, x(t), u, v); \quad u \in P, v \in Q\}$$

under the conditions $x(t_*) \in K$, $t_* \in [t_0, \vartheta_0]$ is uniformly bounded on $[t_*, \vartheta_0]$ by a number $\beta(t_0, K, \vartheta_0)$ for every bounded $K \subset R^n$.

A function $\omega(\vartheta, x, m)$ is given on the set $\{(\vartheta, x, m) : \vartheta \in T, x \in R^n, \ m \in M_0\}$ Here the sets $T \subset [t_0, \vartheta_0]$ and $M = \{(\vartheta, m) : \vartheta \in T, \ m \in M_\vartheta\}$ are assumed to be compact, while the function $\omega(\cdot)$ is assumed to be continuous in the aggregate and continuously differentiable in x in the region $\omega_0 < \omega < \omega^\circ$. The first player, by choosing the control $u \in P$, the instant $\vartheta \in T$, and the point $m \in M_\vartheta$, strives to ensure the inequality $\omega(\vartheta, x[\vartheta], m) \leqslant \varepsilon$, where ε is a given number. The second player chooses a control $v \in Q$ and pursues the opposite goal. Analogously to [2] we identify the first player's strategy U with the function $U(t, x) \subset P$. Every uniform limit of the Euler polygonal lines $x_{\Delta(i)}[t]$ satisfying almost everywhere the equation

$$dx_{\Delta(i)}[t]/dt = f(t, x_{\Delta(i)}[t], u[\tau_k^{(i)}], v[t])$$
$$\tau_{k+1}^{(i)} - \tau_k^{(i)} \leqslant \Delta_i, \quad u[\tau_k^{(i)}] \in U(\tau_k^{(i)}, x_{\Delta(i)}[\tau_k^{(i)}])$$
$$v[t] \in Q \quad \text{is measurable}$$

is called a motion $x[t] = x_U[t] = x[t, t_0, x_0, U]$. Here and below $\{\Delta_i\}$ is a sequence converging to zero $(\Delta_i > 0)$.

The counter-strategy U_v is identified with a function $U_v(t, x, v) \subset P$, while every uniform limit of the Euler polygonal lines $x_{\Delta(i)}[t]$ satisfying almost everywhere the equation

$$dx_{\Delta(i)}[t]/dt = f(t, x_{\Delta(i)}[t], u_t, v[t])$$
$$t \in [\tau_k^{(i)}, \tau_{k+1}^{(i)}], \ \tau_{k+1}^{(i)} - \tau_k^{(i)} \leqslant \Delta_i$$
$$u_t, v[t] \in Q \quad \text{are piecewise-constant}$$
$$u_t \in U_v(\tau_k^{(i)}, x_{\Delta(i)}[\tau_k^{(i)}], v[t])$$

is called a motion $x[t] = x_{U_v}[t] = x[t, t_0, x_0, U_v]$. The second player's strategies V and the motions $x[t] = x_V[t] = x[t, t_0, x_0, V]$ are defined analogously with the natural alterations in notation.

Every uniform limit of the Euler polygonal lines $x_{\Delta(i)}[t]$ satisfying the equation

$$dx_{\Delta(i)}[t]/dt = f(t, x_{\Delta(i)}[t], u[\tau_k^{(i)}], v[\tau_l^{*(i)}])$$
$$u[\tau_k^{(i)}] \in U(\tau_k^{(i)}, \quad x_{\Delta(i)}[\tau_k^{(i)}])$$
$$v[\tau_l^{*(i)}] \in V(\tau_l^{*(i)}, \quad x_{\Delta(i)}[\tau_l^{*(i)}])$$
$$\tau_{k+1}^{(i)} - \tau_k^{(i)} \leqslant \Delta_i, \quad \tau_{l+1}^{*(i)} - \tau_l^{*(i)} \leqslant \Delta_i$$

is called the motion $x[t] = x_{U,V}[t]$ generated by the strategy pair (U, V).

2. Problem 1. Given t_0, x_0 and ϑ_0, for a specified number ε find the counter-strategy U_v° guaranteeing for any motion $x_{U_v^\circ}[t]$ the fulfillment of the relation

$$\omega(\vartheta, x_{U_v^\circ}[\vartheta], m) \leqslant \varepsilon$$

for some $\vartheta \in T$, $m \in M_\vartheta$.

Problem 2. Given t_0, x_0 and ϑ_0, for a specified number ε find the strategy U° guaranteeing for any motion $x_{U^\circ}[t]$ the fulfillment of the relation

$$\omega(\vartheta, x_{U^\circ}[\vartheta], m) \leqslant \varepsilon$$

for some $\vartheta \in T$, $m \in M_\vartheta$.

Problem 2 is solved under the assumption that the saddle point $s'f(\cdot)$ of the "small game" [3] exists. Analogously to [4], we construct a stable system of sets W_ε for solving Problems 1 and 2. Then under the condition that $(t_0, x_0) \in W_\varepsilon$, Problem 1 and, respectively, Problem 2 can be solved by means of counter-strategy U_v° or of strategy U° which, analogously to [5, 6] realize the extremal sighting on some leading motion contained in set W_ε up to the realization of the payoff $\omega \leqslant \varepsilon$ by the first player.

To solve the position game problems posed we use the following program construction. For every $t_* \in [t_0, \vartheta_0]$ and $\vartheta \in [t_*, \vartheta_0]$ we define a class $\{H(m(\cdot)), T_*^{(\vartheta)}\}$ of program controls $\eta(\cdot)$ as the set of all regular Borel measures on $T_*^{(\vartheta)} \times P \times Q$, where $T_*^{(\vartheta)} = [t_*, \vartheta]$, having a Lebesgue projection on $T_*^{(\vartheta)}$: for any Borel set $G \subset T_*^{(\vartheta)}$

$$\eta(G \times P \times Q) = m(G)$$

where $m(\cdot)$ is the Lebesgue measure on a straight line. We identify the class $\{E(m(\cdot)), T_*^{(\vartheta)}\}$ of the second player's program control with the set of all regular Borel measures $\nu(\cdot)$ on $T_*^{(\vartheta)} \times Q$, having a Lebesgue projection on $T_*^{(\vartheta)}$.

For every control $\nu(\cdot) \in \{E(m(\cdot)), T_*^{(\vartheta)}\}$ we define the program $\{H(\nu(\cdot)), T_*^{(\vartheta)}\}$ as the set of all measures $\eta(\cdot) \in \{H(m(\cdot)), T_*^{(\vartheta)}\}$ consistent with measure $\nu(\cdot)$: for any Borel sets $\Delta \subset T_*^{(\vartheta)}$ and $B \subset Q$

$$\eta(\Delta \times P \times B) = \nu(\Delta \times B)$$

Analogously to [7], every absolutely continuous function satisfying almost everywhere the equation

$$dx/dt = \int_P \int_Q f(t, x, u, v)\, \eta_t(du \times dv)$$

where $\eta_t(\cdot)$ is the conditional probability measure [8] corresponding to a given control $\eta(\cdot)$: for any Borel sets $G \subset T_*^{(\vartheta)}$, $A \subset P$ and $B \subset Q$

$$\eta(G \times A \times B) = \int_G m(dt)\, \eta_t(A \times B)$$

where $m(\cdot)$ is the Lebesgue measure on a straight line, is called the program motion $\varphi(t, t_*, x_*, \eta(\cdot))$ for each $\eta(\cdot) \in \{H(m(\cdot)), T_*^{(\vartheta)}\}$ on the interval $T_*^{(\vartheta)}$. It is well known [8] that such a motion exists and is unique for every $\eta(\cdot) \in \{H(m(\cdot)), T_*^{(\vartheta)}\}$.

For the program $\{\Pi(\nu(\cdot)), T_*^{(\vartheta)}\}$ we define by $G(\vartheta, t_*, x_*, \nu(\cdot))$ the attainability region at instant $\vartheta \in [t_*, \vartheta_0]$. This is a compact set in R^n for every measure $\nu(\cdot) \in \{E(m(\cdot)), T_*^{(\vartheta)}\}$. We define the following function:

$$\varepsilon^\circ(t_*, x_*, \vartheta) = \max_{\{E(m(\cdot)),\, T_*^{(\vartheta)}\}} \min_{G(\vartheta, t_*, x_*, \nu(\cdot))} \min_{M_\vartheta} \omega(\vartheta, x, m)$$

$$t_* \in [t_0, \vartheta_0], \qquad \vartheta \in [t_*, \vartheta_0]$$

The control $\eta^\circ(\cdot) \in \{\Pi(\nu(\cdot)), T_*^{(\vartheta)}\}$ is said to be optimal in the program for the instant $\vartheta \in T \cap [t_*, \vartheta_0]$ if

$$\min_{M_\vartheta} \omega(\vartheta, \varphi^\circ(\vartheta), m) = \min_{G(\vartheta, t_*, x_*, \nu(\cdot))} \min_{M_\vartheta} \omega(\vartheta, x, m)$$

$$\varphi^0(\vartheta) = \varphi(\vartheta, t_*, x_*, \eta^\circ(\cdot))$$

By $\{\Pi(\nu(\cdot)), T_*^{(\vartheta)} | t_*, x_*\}_0$ we denote all the program controls optimal in $\{\Pi(\nu(\cdot)), T_*^{(\vartheta)}\}$ for the position (t_*, x_*) and for the instant ϑ. For each $\eta(\cdot) \in \{H(m(\cdot)), T_*^{(\vartheta)}\}$ we form also a set $M_\vartheta^\circ(\eta(\cdot) \mid t_*, x_*)$ as the set of all points $m^\circ \in M_\vartheta$, where $\vartheta \in [t_*, \vartheta_0] \cap T$, for which

$$\omega(\vartheta, \varphi(\vartheta), m^\circ) = \min_{M_\vartheta} \omega(\vartheta, \varphi(\vartheta), m)$$

$$\varphi(\vartheta) = \varphi(\vartheta), t_*, x_*, \eta(\cdot))$$

Every control $\nu^\circ(\cdot) \in \{E(m(\cdot)), T_*^{(\vartheta)}\}$ satisfying the equality

$$\min_{G(\vartheta, t_*, x_*, \nu^\circ(\cdot))} \min_{M_\vartheta} \omega(\vartheta, x, m) = \varepsilon^\circ(t_*, x_*, \vartheta)$$

is called the second player's program control optimal for the instant ϑ and for the position (t_*, x_*). It can be proved that the first and second players' optimal program controls exist for every position (t_*, x_*) and instant $\vartheta \in [t_*, \vartheta_0] \cap T$.

By $o(t_*, x_*, \vartheta)$ we denote the set of all the second player's controls optimal for position (t_*, x_*) and instant ϑ. It can be shown that for a fixed $\vartheta \in [t_*, \vartheta_0] \cap T$ the set $\sigma(t, x, \vartheta)$ is weakly upper semicontinuous by inclusion as the position changes. For a fixed position the function $\varepsilon^\circ(t_*, x_*, \vartheta)$ is lower semicontinuous in ϑ. We denote $\varepsilon^\circ(t_*, x_*) = \min_{[t_*, \vartheta_0] \cap T} \varepsilon^\circ(t_*, x_*, \vartheta)$.

We introduce the set $\Theta(t_*, x_*) \subset [t_*, \vartheta_0] \cap T$ of all instants ϑ_* such that $\varepsilon^\circ(t_*, x_*, \vartheta_*) = \varepsilon^\circ(t_*, x_*)$. For every $\eta^*(\cdot) \in \{H(m(\cdot)), T_*^{(\vartheta)}\}$, $\vartheta \in [t_*, \vartheta_0]$, by $S(\vartheta, t, \varphi^*(\cdot), \eta^*(\cdot))$ we denote the fundamental matrix of solutions of

$$\frac{d\delta x(t)}{dt} = \int_P \int_Q \frac{\partial}{\partial x} f(t, \varphi^*(t), u, v)\, \delta x(t)\, \eta_t^*(du \times dv)$$

Suppose that the control $\eta^\circ(\cdot) \in \{H(m(\cdot)), T_*^{(\vartheta)}\}$, the instant ϑ from the set $[t_*, \vartheta_0] \cap T$ and the point $m^\circ \in M_\vartheta^\circ(\eta^\circ(\cdot) \mid t_*, x_*)$ are such that $\omega_0 < \omega(\vartheta,$

$\varphi^\circ(\vartheta), m^\circ) < \omega^\circ$. Then we form the set $\{l_0 \mid \eta^\circ(\cdot), \vartheta\}$ of all vectors

$$l_0 = \frac{\partial}{\partial x}\omega(\vartheta, \varphi^\circ(\vartheta), m^\circ)$$

as m° ranges over the set $M_\vartheta^\circ(\eta^\circ(\cdot) \mid t_*, x_*)$. Analogously, we define the sets $\{s_0 \mid \eta^\circ(\cdot), \vartheta\}$ and $\{s_0(t) \mid \eta^\circ(\cdot), \vartheta\}$ as sets of all vectors s_0 and functions $s_0(t)$, $t \in T_*^{(\vartheta)}$, described by the relations

$$s_0' = l_0' S(\vartheta, t_*, \varphi^\circ(\cdot), \eta^\circ(\cdot)) \tag{2.1}$$

$$s_0'(t) = l'_0 S(\vartheta, t, \varphi^\circ(\cdot), \eta^\circ(\cdot)) \tag{2.2}$$

respectively, as l_0 ranges over the set $\{l_0 \mid \eta^\circ(\cdot), \vartheta\}$. Then for every $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta)$, $t_* \in [t_0, \vartheta_0]$, $\vartheta \in [t_*, \vartheta_0] \cap T$ and such that $\omega_0 < \varepsilon^\circ(t_*, x_*, \vartheta) < \omega^\circ$, we define the sets

$$S_0(t_*, x_*, \vartheta, \nu^\circ(\cdot)) = \bigcup_{\{\Pi(\nu^\circ(\cdot)),\ T_*^{(\vartheta)} \mid t_*, x_*\}_0} \{s_0 \mid \eta^\circ(\cdot), \vartheta\}$$

$$S_0(t_*, x_*, \vartheta) = \bigcup_{\sigma(t_*, x_*, \vartheta)} S_0(t_*, x_*, \vartheta, \nu(\cdot))$$

We shall assume the fulfillment of the condition: for every position (t_*, x_*) such that $\omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^\circ$ and, here, $t_* \in \Theta(t_*, x_*)$, and for every probability measure $\xi(\cdot)$ on Q we can find an instant $\vartheta_* \in \Theta(t_*, x_*)$ relative to which the following two conditions are fulfilled simultaneously:

Condition 1. There exists a probability measure $\mu_e(\cdot)$ on $P \times Q$, consistent with $\xi(\cdot)$, such that the inequality

$$s_0' \int_P \int_Q f(t_*, x_*, u, v)\,\mu_e(du \times dv) \leqslant \max_Q \min_P s_0' f(t_*, x_*, u, v)$$

is fulfilled for each $s_0 \in S_0(t_*, x_*, \vartheta_*)$

Condition 2. For any program control $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta_*)$ and any $\eta^\circ(\cdot) \in \{\Pi(\nu^\circ(\cdot)),\ T_*^{(\vartheta)} \mid t_*, x_*\}_0$ the following maximum condition (Δ is any Borel subset of $T_*^{(\vartheta)}$) is fulfilled for each function $s_0(t) \in \{s_0(t) \mid \eta^\circ(\cdot), \vartheta_*\}$:

$$\int_\Delta \int_P \int_Q s_0'(t) f(t, \varphi^\circ(t), u, v)\,\eta^\circ(dt \times du \times dv) =$$

$$\int_\Delta \max_Q \min_P [s_0'(t) f(t, \varphi^\circ(t), u, v)]\,m(dt)$$

We note that every $\eta^\circ(\cdot) \in \{\Pi(\nu^\circ(\cdot)),\ T_*^{(\vartheta)} \mid t_*, x_*\}_0$, where $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta_*)$, satisfies the minimum condition

$$\int_\Delta \int_P \int_Q s_0'(t) f(t, \varphi^\circ(t), u, v)\,\eta^\circ(dt \times du \times dv) =$$

$$\int_\Delta \int_Q \min_P [s_0'(t) f(t, \varphi^\circ(t), u, v)]\,\nu^\circ(dt \times dv)$$

on any Borel set $\Delta \subset T_*^{(\vartheta)}$ for every $s_0(t) \in \{s_0(t) \mid \eta^\circ(\cdot)\ \vartheta_*\}$.

3. Suppose that a position (t_*, x_*) and an instant ϑ such that $t_* \in [t_0, \vartheta_0)$, $\vartheta \in [t_*, \vartheta_0] \cap T$ and $\omega_0 < \varepsilon^\circ(t_*, x_*, \vartheta) < \omega^\circ$ and a position (t, x) contained in a sufficiently small neighborhood of (t_*, x_*), where $t \in T_*^{(\vartheta)}$, have been chosen.

Arbitrarily we choose the second player's controls $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta)$ and $\nu_0(\cdot)^\Delta \in \sigma(t, x, \vartheta)$ and we construct the pasted-together program control $\nu_0(\cdot)_0^\Delta$ obtained by replacing the measure $\nu^\circ(\cdot)$ by $\nu_0(\cdot)^\Delta$ on $[t, \vartheta] \times Q$. Let

$$\eta_0(\cdot)_0^\Delta \in \{\Pi(\nu_0(\cdot)_0^\Delta), T_*^{(\vartheta)} \mid t_*, x_*\}_0$$
$$m_{00}^\Delta \in M_\vartheta^\circ(\eta_0(\cdot)_0^\Delta \mid t_*, x_*), \quad \bar\varphi_0(\vartheta)_0^\Delta = \varphi(\vartheta, t, x_*, \bar\eta_0(\cdot)_0^\Delta)$$

where $\bar\eta_0(\cdot)_0^\Delta$ is the measure $\eta_0(\cdot)_0^\Delta$ considered on $[t, \vartheta] \times P \times Q$. Then by $\{l_0^* \mid \bar\eta_0(\cdot)_0^\Delta, \vartheta\}$ we denote the set of all vectors

$$l_0^* = \frac{\partial}{\partial x}\omega(\vartheta, \bar\varphi_0(\vartheta)_0^\Delta, m_{00}^\Delta) \tag{3.1}$$

as m_{00}^Δ ranges over $M_\vartheta^\circ(\eta_0(\cdot)_0^\Delta \mid t_*, x_*)$, while $\{s_0^* \mid \bar\eta_0(\cdot)_0^\Delta, \vartheta\}$ is the set of all vectors of the form

$$s_0^{*\prime} = l_0^{*\prime} S(\vartheta, t, \bar\varphi_0(\cdot)_0^\Delta, \bar\eta_0(\cdot)_0^\Delta) \tag{3.2}$$

as l_0^* ranges over $\{l_0^* \mid \bar\eta_0(\cdot)_0^\Delta, \vartheta\}$. We introduce the set

$$S_0^*(t, x_*, \Delta\mathrm{x} \mid \nu^\circ(\cdot), \nu_0(\cdot)^\Delta, \vartheta) = \bigcup_{\{\Pi(\nu_0(\cdot)_0^\Delta), T_*^{(\vartheta)} \mid t_*, x_*\}_0} \{s_0^* \mid \bar\eta_0(\cdot)_0^\Delta, \vartheta\}$$

Lemma 3.1. Let (t_*, x_*) and the instant $\vartheta \in (t_*, \vartheta_0] \cap T$ be such that $\omega_0 < \varepsilon^\circ(t_*, x_*, \vartheta) < \omega^\circ$. Then for any $\alpha > 0$ we can find $\delta(\alpha, t_*, x_*, \vartheta) > 0$ such that for every position $(t, x): 0 \leqslant t - t_* < \delta(\alpha, t_*, x_*, \vartheta)$, $\|x - x_*\| < \delta(\alpha, t_*, x_*, \vartheta)$, $\|\cdot\|$ is the Euclidean norm, we can find program controls $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta)$ and $\nu_0(\cdot)^\Delta \in \sigma(t, x, \vartheta)$ such that vectors

$$s_0 \in S_0(t_*, x_*, \vartheta, \nu^\circ(\cdot)) \tag{3.3}$$
$$s_0^* \in S_0^*(t, x_*, \Delta\mathrm{x} \mid \nu^\circ(\cdot), \nu_0(\cdot)^\Delta, \vartheta)$$

exist, for which the inequality

$$\|s_0 - s_0^*\| < \alpha \tag{3.4}$$

is fulfilled.

Proof. We assume the contrary. Then there exist a position (t_*, x_*) and an instant ϑ, satisfying the lemma's hypotheses, a number $\alpha > 0$, and a sequence $\{(t_n, x_n)\}$ of positions converging to (t_*, x_*), such that for each $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta)$ and $\nu_0(\cdot)^\Delta \in \sigma(t_n, x_n, \vartheta)$ we have, for all n,

$$\|s_0 - s_0^*\| \geqslant \alpha \tag{3.5}$$

for any s_0 and s_0^* satisfying inclusions (3.3). Then we choose $\nu_n^\circ(\cdot) \in \sigma(t_n, x_n, \vartheta)$, considering without loss of generality that $\{\nu_n^\circ(\cdot)\}$ converges weakly to some $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta)$, and we construct the controls $\nu_0(\cdot)_0^n$ pasted together from $\nu^\circ(\cdot)$ and $\nu_n^\circ(\cdot)$, we choose $\eta_0(\cdot)_0^n \in \{\Pi(\nu_0(\cdot)_0^n), T_*^{(\vartheta)} \mid t_*, x_*\}_0$ and $m_{00}^n \in M_\vartheta^\circ(\eta_0(\cdot)_0^n \mid t_*, x_*)$ which once again can be considered as converging weakly, respectively, to $\eta^\circ(\cdot) \in \{\Pi(\nu^\circ(\cdot)), T_*^{(\vartheta)} \mid t_*, x_*\}_0$ and $m^\circ \in M_\vartheta^\circ(\eta_0(\cdot) \mid t_*, x_*)$, and we construct the vectors $s_{0,n}^*$ with the aid of (3.2) for $\eta_0(\cdot)_0^\Delta = \eta_0(\cdot)_0^n$ and $m_{00}^\Delta = m_{00}^n$ and the vector s_0 defined by (2.1) for the resulting $\eta^\circ(\cdot)$ and m°. From the weak convergence of $\{\eta_0(\cdot)_0^n\}$ follows the uniform convergence of the fundamental matrices $S(\vartheta, t, \varphi_0(\cdot)_0^n, \eta_0(\cdot)^n)$, where $\varphi_0(t)_0^n = \varphi(t, t_*, x_*, \eta_0(\cdot)_0^n)$, to the fundamental matrix $S(\vartheta, t, \varphi^0(\cdot), \eta^\circ(\cdot))$. Hence we can show that $s_{0,n}^* \to s_0$, which contradicts (3.5).

Lemma 3.2. Let $(t_*, x_*) : t_* \in \Theta$ and the instant $\vartheta \in \Theta = \Theta(t_*, x_*)$ be such that $\omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^\circ$ and, moreover, let Conditions 1 and 2 be fulfilled simultaneously for an instant ϑ for a chosen probability measure $\xi(\cdot)$. Then for any $\gamma > 0$ there exists $\delta = \delta(\gamma, t_*, x_*, \vartheta) > 0$ such that the relation

$$\Delta\varepsilon^\circ = \varepsilon^\circ[t] - \varepsilon^\circ[t_*] < \gamma(t - t_*) \tag{3.6}$$

is fulfilled for every $t \in [t_*, t_* + \delta)$. Here

$$\varepsilon^\circ[t] = \varepsilon^\circ(t, \varphi_e(t))$$

$$\varphi_e(t) = \varphi(t, t_*, x_*, \eta_e(\cdot))$$

$$\eta_e(\cdot): \eta_e(G \times A \times B) = m(G)\mu_e(A \times B)$$

for any Borel sets $G \subset T_*^{(\vartheta)}$, $A \subset P$ and $B \subset Q$; for a given ϑ, $\mu_e(\cdot)$ satisfies Condition 1 in the class of probability measures consistent with $\xi(\cdot)$.

Proof. Setting $x = \varphi_e(t)$, we estimate the increment of the function $\varepsilon^\circ(t, x)$ along the motion $\varphi_e(t)$. Choosing

$$\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta),\ \nu_0(\cdot)^\Delta \in \sigma(t, x, \vartheta)$$
$$\eta_0(\cdot)_0^\Delta \in \{\Pi(\nu_0(\cdot)_0^\Delta),\ T_*^{(\vartheta)} \mid t_*, x_*\}_0,\ m_{00}^\Delta \in M_\vartheta^\circ(\eta_0(\cdot)_0^\Delta \mid t_*, x_*)$$

arbitrarily, we obtain

$$\Delta\varepsilon^\circ \leqslant \omega(\vartheta, \varphi(\vartheta, t, x, \bar{\eta}_0(\cdot)_0^\Delta), m_{00}^\Delta) - \omega(\vartheta, \varphi(\vartheta, t_*, x_*, \eta_0(\cdot)_0^\Delta), m_{00}^\Delta) \tag{3.7}$$

Here $\bar{\nu}_0(\cdot)_0^\Delta$ is the second player's program control pasted together from ν° and $\nu_0(\cdot)^\Delta$. It can be shown that for any $\alpha > 0$ we can find $\delta_\alpha = \delta_\alpha(t_*, x_*, \vartheta) > 0$ such that for each $(t, x) : 0 \leqslant t - t_* < \delta_\alpha$, $\|x - x_*\| < \delta_\alpha$ the inequality

$$|\omega(\vartheta, \bar{\varphi}_0(\vartheta)_0^\Delta, m_{00}^\Delta) - \varepsilon^\circ(t_*, x_*)| < \alpha \tag{3.8}$$

is fulfilled uniformly with respect to

$$\nu^\circ(\cdot) \in \sigma(t_*, x^*, \vartheta),\ \nu_0(\cdot)^\Delta \in \sigma(t, x, \vartheta)$$
$$\eta_0(\cdot)_0^\Delta \in \{\Pi(\nu_0(\cdot)_0^\Delta),\ T_*^{(\vartheta)} \mid t_*, x_*\}_0,\ m_{00}^\Delta \in M_\vartheta^\circ(\eta_0(\cdot)_0^\Delta \mid t_*, x_*)$$

Taking into account the uniform boundedness and the property of differentiability with respect to x of the function $\omega(\vartheta, x, m)$, with due regard to (3.7) and (3.8) we can show that

$$\Delta\varepsilon^\circ \leqslant l_0^{*\prime} S(\vartheta, t, \bar{\varphi}_0(\cdot)_0^\Delta, \bar{\eta}_0(\cdot)_0^\Delta)(\Delta x - \Delta\varphi_{00}^\Delta) + o(\Delta t)$$

where

$$l_0^* = \frac{\partial}{\partial x}\omega(\vartheta, \bar{\varphi}_0(\vartheta)_0^\Delta, m_{00}^\Delta)$$

$$\Delta\varphi_{00}^\Delta = \int_{t_*}^{t}\int_P\int_Q f(\tau, \varphi_0(\tau)_0^\Delta, u, v)\,\eta_0(d\tau \times du \times dv)_0^\Delta$$

Using Lemma 3.1, for each position (t, x) from a sufficiently small neighborhood of (t_*, x_*) we can choose a pair of program controls $\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta)$ and $\nu_0(\cdot)^\Delta \in \sigma(t, x, \vartheta)$ of the second player such that we find

$$s_0 \in S_0(t_*, x_*, \vartheta, \nu^\circ(\cdot)),\ s_0^* \in S_0^*(t, x_*, \Delta x \mid \nu^\circ(\cdot), \nu_0(\cdot)^\Delta, \vartheta)$$

satisfying (3.4); moreover, such a choice is possible for any $\alpha > 0$. We obtain the vector s_0^* by choosing suitable

$$\eta_0(\cdot)_0^{\Delta} \in \{\Pi(\nu_0(\cdot)_0^{\Delta}),\ T_*^{(\vartheta)} \mid t_*, x_*\}_0,\quad m_{00}^{\Delta} \in M\vartheta^{\circ}(\eta_0(\cdot)_0^{\Delta} \mid t_*, x_*)$$

Then after some manipulations with due regard to Condition 2 we obtain

$$\Delta\varepsilon^{\circ} \leqslant \Big[s_0' \int\limits_P \int\limits_Q f(t_*, x_*, u, v)\,\mu_e(du \times dv) - \max_Q \min_P s_0' f(t_*, x_*, u, v) \Big] \Delta t + o(\Delta t)$$

We obtain the lemma's assertion by using Condition 1.

4. For each number γ we denote by W_γ the set of all positions $(t, x), t \in [t_0, \vartheta_0]$, for which $\varepsilon^{\circ}(t, x) \leqslant \gamma$.

Lemma 4.1. At each position (t_*, x_*), $t_* \in [t_0, \vartheta_0]$, for each number $\alpha > 0$ we can find $\delta > 0$ such that for all $(t, x) : |t - t_*| < \delta$, $\|x - x_*\| < \delta$, the inequality

$$|\varepsilon^{\circ}(t, x, \vartheta) - \varepsilon^{\circ}(t_*, x_*, \vartheta)| < \alpha$$

is fulfilled uniformly with respect to $\vartheta \in [\max(t, t_*), \vartheta_0] \cap T$.

It can be shown that set W_γ is closed for each γ.

Lemma 4.2. One of the following two statements is true concerning any position (t_*, x_*), $t_* \in [t_0, \vartheta_0]$:

a) the instant $t_* \in \Theta(t_*, x_*)$;

b) for any $\alpha > 0$ there exists $\delta > 0$ such that

$$|\varepsilon^{\circ}(t, x) - \varepsilon^{\circ}(t_*, x_*)| < \alpha \quad \text{for } 0 \leqslant t - t_* < \delta, \quad \|x - x_*| < \delta$$

for each (t, x).

Proof. Suppose $t_* \overline{\in} \Theta(t_*, x_*)$. Using the lower semicontinuity of $\varepsilon^{\circ}(t_*, x_*, \vartheta)$ with respect to ϑ, we can show that $\Theta(t_*, x_*)$ is closed. Then $\chi > 0$ exists such that $[t_*, t_* + \chi) \cap \Theta(t_*, x_*) = \phi$. With due regard to Lemma 4.1, for any $\alpha > 0$ we can find $\delta, 0 < \delta < \chi$ such that $|\varepsilon^{\circ}(t, x, \vartheta) - \varepsilon^{\circ}(t_*, x_*, \vartheta)| < \alpha$ uniformly with respect to $\vartheta \in [t, \vartheta_0] \cap T$ for all $(t, x) : 0 \leqslant t - t_* < \delta$, $\|x - x_*\| < \delta$; whence follows satatement (b) of the lemma, with due regard to the fact that $\Theta(t_*, x_*) \subset [t, \vartheta_0] \cap T$.

Lemma 4.3. For any position $(t_*, x_*) \in W_\gamma$, $\omega_0 < \gamma < \omega^{\circ}$, for every number $\alpha > 0$ not exceeding some $\alpha_0 > 0$, for any instant $t^* \in [t_*, \vartheta_0]$ and any probability measure $\xi(\cdot)$ on Q, one of the following statements is true, relative to the family of program motions $X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ generated by the program $\{\Pi(\nu_\xi(\cdot)), \Delta\}$, where $\Delta = [t_*, t^*]$ and $\nu_\xi(G \times B) = m(G)\,\xi(B)$ for any Borel sets $G \subset \Delta$ and $B \subset Q$:

a) there exists $\varphi_\alpha(t) \in X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ and an instant $\vartheta_\alpha \in \Delta \cap T$ such that

$$\min_{M_{\vartheta_\alpha}} \omega(\vartheta_\alpha, \varphi_\alpha(\vartheta_\alpha), m) \leqslant \gamma + \alpha(\vartheta_\alpha - t_*)$$

b) there exists $\varphi_\alpha^{\circ}(t) \in X_\Delta(t_*, x_*\ \nu_\xi(\cdot))$ such that $(t, \varphi_\alpha^{\circ}(t)) \in W_{\gamma + \alpha(t - t_*)}$ for all $t \in \Delta$.

Proof. We choose α_0 as follows: let $\beta > 0$ and $\gamma < \omega^{\circ} - \beta$, then $\alpha_0 = \beta/(\vartheta_0 - t_0)$. Let us assume that the lemma is incorrect for the given α_0. Then we can find a position $(t_*, x_*) \in W_\gamma$, an instant t^*, and a probability measure $\xi(\cdot)$ such that statements (a) and (b) of the lemma are simultaneously violated for some $\alpha : 0 < \alpha < \alpha_0$. This signifies that for any $\varphi(t) \in X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ we can find the first instant τ_φ: $t_* < \tau_\varphi < t^*$ such that for any $\delta > 0$ in the semi-interval $[\tau_\varphi, \tau_\varphi + \delta)$ we can find

t_δ such that $\varepsilon^\circ(t_\delta, \varphi(t_\delta)) > \gamma + \alpha(t_\delta - t_*)$, because otherwise statement (b) would be fulfilled. Because $W_{\gamma+\alpha(\tau_\varphi - t_*)}$ is closed the position $(\tau_\varphi, \varphi(\tau_\varphi))$ is contained in $W_{\gamma+\alpha(\tau_\varphi - t_*)}$. Let $\tau_\xi = \max \tau_\varphi$ in the class $\varphi(t) \in X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ and let $\bar\varphi(t)$ be the motion from $X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ realizing τ_ξ. By assumption $\tau_\xi \overline{\in} \Theta(\tau_\xi, \bar\varphi(\tau_\xi))$ and by virtue of Lemma 4.2, $\omega_0 < \varepsilon^\circ(\tau_\xi, \bar\varphi(\tau_\xi)) = \gamma + \alpha(\tau_\xi - t_*) < \omega^\circ$. Then we can find an instant $\bar\vartheta \in \Theta(\tau_\xi, \bar\varphi(\tau_\xi))$ relative to which Conditions 1 and 2 are fulfilled simultaneously and, hence, so is Lemma 3.2, taking which into account we arrive at a contradiction with the fact that $\tau_\xi = \max \tau_\varphi$ on $X_\Delta(t_*, x_*, \nu_\xi(\cdot))$.

Lemma 4.4. For any $\gamma : \omega_0 \leqslant \gamma < \omega^\circ$ the set W_γ is u-stable for any position $(t_*, x_*) \in W_\gamma$, any instant $t^* \in [t_*, \vartheta_0]$ and any probability measure $\xi(\cdot)$ on Q, one of the following two statements is valid for the family $X_\Delta(t_*, x_*, \nu_\xi(\cdot))$:

1) there exist $\varphi(t) \in X_\Delta(t_*, X_*, \nu_\xi(\cdot))$ and an anstant $\vartheta_* \in \Delta \cap T$ such that

$$\min_{M_{\vartheta_*}} \omega(\vartheta_*, \varphi(\vartheta_*), m) \leqslant \gamma$$

2) there exists $\varphi(t) \in X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ such that $(t, \varphi(t)) \in W_\gamma$ for any $t \in \Delta$.

The proof follows from Lemma 4.3 and from the fact that the family $X_\Delta(t_*, x_*, \nu_\xi(\cdot))$ is compact in itself.

The following theorems are proved analogously as in [4 – 6].

Theorem 4.1. Let $\omega_0 \leqslant \varepsilon = \varepsilon^\circ(t_0, x_0) < \omega^\circ$. Then the counter-strategy U_r° realizing the extremal sighting on the leading motion solves Problem 1.

Theorem 4.2. Let $\omega_0 \leqslant \varepsilon = \varepsilon^\circ(t_0, x_0) < \omega^\circ$ and let a saddle point exist in in the small game. Then the strategy U° realizing the extremal sighting on a leading motion solves Problem 2.

Let $\omega(\vartheta, x, m) = \| x - m \|$ for all $\vartheta \in [t_0, \vartheta_0]$, where $m \in M_*$ and M_* is a closed subset of R^n. Then Problems 1 and 2 are problems of position encounter with set M_* by instant ϑ_0. The possibility that in the given case M_* is noncompact is unessential because the problem reduces to the problem of encounter with some compact subset of M_*.

5. Problem 3. Given (t_0, x_0), $\vartheta_0 > t_0$ and ε, find strategy V° guaranteeing

$$\min_T \min_{M_\vartheta} \omega(\vartheta, x_{V^\circ}[\vartheta], m) \geqslant \varepsilon$$

for every motion $x_{V^\circ}[t]$.

Problem 4. Construct a strategy pair (U°, V°) such that the inequality

$$\sup_{\{x_{U^\circ, V}[t]\}} \min_T \min_{M_\vartheta} \omega(\vartheta, x_{U^\circ, V}[\vartheta], m) \leqslant$$

$$\min_T \min_{M_\vartheta} \omega(\vartheta, x^\circ[\vartheta], m) \leqslant \inf_{\{x_{U, V^\circ}[t]\}} \min_T \min_{M_\vartheta} \omega(\vartheta, x_{U, V^\circ}[\vartheta], m)$$

is fulfilled for every motion $x^\circ[t] = x_{U^\circ, V^\circ}[t]$.

It is well known that to solve Problem 3 it suffices to construct a v-stable [3] system of sets and to choose as V^0 the strategy realizing the extremal sighting on some leading motion [6] maintained in this system of sets. Problem 4 is successfully solved if Conditions 1, 2 are fulfilled together with the conditions sufficient for solving Problem 3. In case M_* is a closed set in space R^n, while $\omega(\vartheta, x, m) = \| x - m \|$, Problem 3 is the usual evasion problem [3] and Problem 4, respectively, is the encounter-evasion problem with the target set M_*.

Lemma 5.1. For every position (t_*, x_*) the function $\varepsilon^\circ(t, x) = \min_{[t, \vartheta_0] \cap T} \varepsilon^\circ$

(t, x, ϑ) satisfies the following condition: for any number $\alpha > 0$ there exists $\delta = \delta(\alpha, t_*, x_*) > 0$ such that

$$\varepsilon^\circ(t_*, x_*) < \alpha + \varepsilon^\circ(t, x)$$

for every position $(t, x) : 0 \leqslant t - t_* < \delta, \ \| x - x_* \| < \delta$.

Lemma 5.2. For every position (t_*, x_*) and any number $\alpha > 0$ we can find $\delta_\alpha = \delta_\alpha(t_*, x_*) > 0$ such that

$$\Theta(t, x) \subset \Theta^\alpha(t_*, x_*),$$

where Θ^α is the α-neighborhood of set Θ , for any position $(t, x) : 0 \leqslant t - t_* < \delta_\alpha, \ \| x - x_* \| < \delta_\alpha, \ \varepsilon^\circ(t, x) \leqslant \varepsilon^\circ(t_*, x_*)$.

The lemma is proved by contradiction, using Lemma 5.1. The condition $\varepsilon^\circ(t, x) \leqslant \varepsilon^0(t_*, x_*)$ is highly essential. Indeed, if we examine the scalar system

$$dx/dt = u - v$$
$$\omega(\vartheta, x, m) = | x - m | \quad \text{for} \quad \vartheta = \vartheta_1 \text{ and } \vartheta = \vartheta_2$$

and the set $M = \{\vartheta_1, M_{\vartheta_1}\} \cup \{\vartheta_2, M_{\vartheta_2}\}$, where $\vartheta_1 < \vartheta_2, \quad M_{\vartheta_1} = [0, a], \quad M_{\vartheta_2} = \{0\}$, $| u | \leqslant \mu, \ | v | \leqslant \nu$, and, moreover, $\mu + \nu < a / (\vartheta_2 - \vartheta_1)$, then the set $\Theta(\vartheta_1, a) = \{\vartheta_1\}$ for the position (ϑ_1, a) , while $\Theta(t, x) = \{\vartheta_2\}$ for every position (t, x) along the motion from position (ϑ_1, a) for $t > \vartheta_1$. Note that for a fixed position (t_*, x_*) the set $\sigma(t_*, x_*, \vartheta)$ cannot possess, as ϑ varies, the property of weak upper semicontinuity by inclusion. For example, in the system

$$dx / dt = u - v, \ \ | u | \leqslant \mu, \ \ | v | \leqslant \nu, \ \ t \in [0, \vartheta_0], \ \nu > \mu$$
$$M_\vartheta = \phi \quad \text{for} \quad \vartheta < \vartheta_1 < \vartheta_0$$
$$M_{\vartheta_1} = \{x : x \leqslant 0\}, \ M = \{k(\vartheta - \vartheta_1)\} \ \text{for} \ \ \vartheta > \vartheta_1, \ k > 0$$

for the position $(t_0 = 0, x_0 = 0)$ the second player's optimal program control relative to the instant $\vartheta > \vartheta_1$ is $v_{\text{opt}}(t) = + \nu$ almost everywhere and is unique, while relative to the instant $\vartheta = \vartheta_1$ it is $v_{\text{opt}}(t) = -\nu$ almost everywhere and also is unique; note that at the point ϑ_1 the function

$$\varepsilon^0(t_0, x_0, \vartheta) = (\nu - \mu)\vartheta + k(\vartheta - \vartheta_1), \ \vartheta \in [\vartheta_1, \vartheta_0]$$

is right-continuous in ϑ.

6. We make the following auxiliary constructions: together with a position (t_*, x_*) we consider a neighboring position $(t, x), \ t \geqslant t_*$. Let $\vartheta \in \Theta(t, x), \ \nu_0(\cdot) \in \sigma(t_*, x_*, \vartheta)$, let measure $\bar{\nu}_0(\cdot) \in \{E(m(\cdot)), [t, \vartheta]\}$ coincide with $\nu_0(\cdot)$ on $[t, \vartheta] \times Q$, let $\bar{\eta}_0(\cdot) \in \{\Pi(\bar{\nu}_0(\cdot)), [t, \vartheta] \mid t, x\}_0, \ \bar{m}_0 \in M_\vartheta^0(\bar{\eta}_0(\cdot) \mid t, x)$. By $S^*(t, x, \mid t_*, x_*)$ we denote the set of all vectors s_* of the form

$$s_*' = \left[\frac{\partial}{\partial x}\omega(\vartheta, \bar{\varphi}_0(\vartheta), \bar{m}_0)\right]' S(\vartheta, t, \bar{\varphi}_0(\cdot), \bar{\eta}_0(\cdot)) \tag{6.1}$$

as ϑ ranges over $\Theta(t, x), \ \nu_0(\cdot)$ ranges over $\sigma(t_*, x_*, \vartheta), \ \bar{\eta}_0(\cdot)$ ranges over $\{\Pi(\bar{\nu}_0(\cdot)), [t, \vartheta] \mid t, x\}_0$ and $\bar{m}_0$ ranges over $M_\vartheta^0(\bar{\eta}_0(\cdot) \mid t, x)$. Such sets can be constructed for all positions $(t, x), t \geqslant t_*, \varepsilon^\circ(t, x) \leqslant \varepsilon^\circ(t_*, x_*)$, from some neighborhood of $(t_*, x_*), \ \omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^\circ$.

We assume that the following condition is fulfilled.

Condition 3. For every position (t_*, x_*) such that $\omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^\circ$ and for a probability measure $\mu(\cdot)$ on P we can find a probability measure $\xi(\cdot)$

on $P \times Q$ consistent with $\mu(\cdot)$ and such that for any number $\alpha > 0$ there exists $\delta_\alpha = \delta_\alpha(t_*, x_*, \xi(\cdot)) > 0$ along the program motion $\varphi_\xi(t) = \varphi(t, t_*, x_*, \eta_\xi(\cdot))$ and that for every position

$$(t, \varphi_\xi(t)),\ t \in [t_*, t_* + \delta_\alpha), \quad \varepsilon^\circ(t, \varphi_\xi(t)) \leqslant \varepsilon^\circ(t_*, x_*)$$

we can find a vector $s_* \in S^*(t, \varphi_\xi(t) \mid t_*, x_*)$, satisfying the condition

$$s_*' \int_P \int_Q f(t_*, x_*, u, v)\, \xi(du \times dv) \geqslant \max_Q \min_P s_*' f(t_*, x_*, u, v) - \alpha$$

Here $\eta_\xi(\cdot)$ is a program control for which $\eta_{\xi,t}(\cdot) = \xi(\cdot)$. Let us present more intuitive conditions under whose fulfillment Condition 3 is satisfied.

Condition 4. The sets $\sigma(t_*, x_*, \vartheta)$ are weakly upper semicontinuous by inclusion at each point $\vartheta_* \in \Theta(t_*, x_*)$ for every position (t_*, x_*) such that $\omega_0 < \varepsilon^0(t_*, x_*) < \omega^0$.

This condition is always fulfilled when M_ϑ varies continuously with varying ϑ, when

$$M = \bigcup_1^{k_0} \{\vartheta_k, M_{\vartheta_k}\}$$

and, in a large number of other cases, when M_ϑ does not possess the property of continuity in ϑ.

Condition 5. For every position $(\omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^0)$, for any probability measure $\mu(\cdot)$ on P we can find a probability measure $\xi(\cdot)$ on $P \times Q$, consistent with it, for which the inequality

$$s_0' \int_P \int_Q f(t_*, x_*, u, v)\, \xi(du \times dv) \geqslant \max_Q \min_P s_0' f(t_*, x_*, u, v) \qquad (6.2)$$

is fulfilled on each vector

$$s_0 \in \bigcup_{\Theta(t_*, x_*)} S_0(t_*, x_*, \vartheta) = S_0(t_*, x_*)$$

Lemma 6.1. Condition 3 is always fulfilled when Conditions 4 and 5 are fulfilled.

Proof. We choose any position (t_*, x_*) such that $\omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^\circ$ and for neighboring positions $\leftrightarrow (t, x)$, $t \geqslant t_*$ and $\varepsilon^\circ(t, x) \leqslant \varepsilon^\circ(t_*, x_*)$, we examine the sets $S^*(t, x \mid t_*, x_*)$. Let us show that for any $\alpha > 0$ we can find $\delta_\alpha = \delta_\alpha(t_*, x_*) > 0$ such that

$$S^*(t, x \mid t_*, x_*) \subset S_0{}^\alpha(t_*, x_*)$$

for every position (t, x): $0 \leqslant t - t_* < \delta_\alpha$, $\| x - x_* \| < \delta_\alpha$, $\varepsilon^\circ(t, x) \leqslant \varepsilon^\circ(t_*, x_*)$. In fact, let us assume the contrary. Then we can find $\{(t_n, x_n)\}$ satisfying the above-stated conditions and such that $(t_n, x_n) \to (t_*, x_*)$; moreover, for each n there exists $s_n \in S^*(t_n, x_n \mid t_*, x_*)$ such that

$$\| s_n - s_0 \| \geqslant \alpha \qquad (6.3)$$

for every $s_0 \in S_0(t_*, x_*)$. This signifies that

$$\begin{gathered} \{\vartheta_n\} : \vartheta_n \in \Theta(t_n, x_n), \qquad \nu_n{}^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta_n) \\ \bar\eta_n{}^\circ(\cdot) \in \{\Pi(\nu_n{}^\circ(\cdot)),\ [t_n, \vartheta_n] \mid t_n, x_n\}_0 \\ \bar m_n{}^\circ \in M_{\vartheta_n}{}^\circ(\bar\eta_n{}^\circ(\cdot) \mid t_n, x_n) \end{gathered}$$

exist for which s_n is expressed by (6.1).

Without loss of generality we can assume that the sequences $\{\vartheta_n\}$ and $\{\bar m_n{}^\circ\}$ converge

$$\vartheta_n \to \vartheta_* \in \Theta(t_*, x_*), \qquad \overline{m}_n^\circ \to m^\circ \in M_{\vartheta_*}$$

and that the sequences $\{\nu_n^\circ(\cdot)\}$ and $\{\overline{\eta}_n^\circ(\cdot)\}$ converge weakly, respectively, to the program controls

$$\nu^\circ(\cdot) \in \sigma(t_*, x_*, \vartheta_*), \qquad \eta^\circ(\cdot) \in \{\Pi(\nu^\circ(\cdot)), [t_*, \vartheta_*]\}$$

Then,

$$\omega(\vartheta_*, \varphi^\circ(\vartheta_*), m^\circ) = \lim_n \omega(\vartheta_n, \overline{\varphi}_n^\circ(\vartheta_n), \overline{m}_n^\circ) =$$
$$\lim_n \min_{G(\vartheta_n, t_n, x_n, \overline{\nu}_n^\circ(\cdot))} \min_{M_{\vartheta_n}} \omega(\vartheta_n, x, m)$$

On the other hand, it can be shown that

$$\lim_n \omega(\vartheta_n, \overline{\varphi}_n^\circ(\vartheta_n), \overline{m}_n^\circ) = \varepsilon^\circ(t_*, x_*)$$

Hence

$$\omega(\vartheta_*, \varphi^\circ(\vartheta_*), m^\circ) = \min_{G(\vartheta_*, t_*, x_*, \nu^\circ(\cdot))} \min_{M_{\vartheta_*}} \omega(\vartheta_*, x, m)$$

Thus,

$$\eta^\circ(\cdot) \in \{\Pi(\nu^\circ(\cdot)), [t_*, \vartheta_*] \mid t_*, x_*\}_0 \tag{6.4}$$
$$m^\circ \in \overset{\circ}{M}_{\vartheta_*}(\eta^\circ(\cdot) \mid t_*, x_*)$$

Then, a vector s_0 of the form

$$s_0' = \left[\frac{\partial}{\partial x}\omega(\vartheta_*, \varphi^\circ(\vartheta_*), m^\circ)\right]' S(\vartheta_*, t_*, \varphi^\circ(\cdot), \eta^\circ(\cdot))$$

belongs to $S_0(t_*, x_*)$. From the weak convergence of $\{\overline{\eta}_n^\circ(\cdot)\}$ to $\eta^\circ(\cdot)$, as well as with due regard to $t_n \to t_*$, $\vartheta_n \to \vartheta_*$ we obtain

$$\lim_n \| S(\vartheta_n, t_n, \overline{\varphi}_n^\circ(\cdot), \overline{\eta}_n^\circ(\cdot)) - S(\vartheta_*, t_*, \varphi^\circ(\cdot), \eta^\circ(\cdot)) \| = 0$$

But then $s_0 = \lim_n s_n$, which contradicts (6.3). Hence follows the validity of Condition 3.

We note that $S^*(t, x \mid t_*, x_*) = S_0(t_*, x_*)$ for $t = t_*$, $x = x_*$.

The fulfillment of Condition 4 is highly essential because in the general case the program control $\nu^0(\cdot)$ may not belong to the set $\sigma(t_*, x_*, \vartheta_*)$, as we see from the example of the linear system $dx / dt = u - v$ analyzed in Sect. 5, as a result of which condition (6.2) may not be fulfilled on a vector s_0 constructed by means of (6.4). We can waive Condition 4 if we require the fulfillment of (6.2) on a properly augmented set of vectors s_0.

Lemma 6.2. For any position, $\omega_0 < \varepsilon^\circ(t_*, x_*) < \omega^\circ$, and for a probability measure $\mu(\cdot)$ on P, the bound $\varepsilon^\circ(t, \varphi_\xi(t)) - \varepsilon^\circ(t_*, x_*) \geqslant o(t - t_*) = o(\Delta t)$, where $o(t - t_*) / (t - t_*) \to 0$ as $t \to t_*$, is fulfilled along the motion $\varphi_\xi(t) = \varphi(t, t_*, x_*, \eta_\xi(\cdot))$, where $\xi(\cdot)$ has been chosen from Condition 3.

Proof. Set $x = \varphi_\xi(t)$ and let $\varepsilon^\circ(t, x) \leqslant \varepsilon^\circ(t_*, x_*)$. For every

$$\vartheta \in \Theta(t, x), \ \nu_0(\cdot) \in \sigma(t_*, x_*, \vartheta)$$
$$\overline{\eta}_0(\cdot) \in \{\Pi(\overline{\nu}_0(\cdot)), [t, \vartheta] \mid t, x\}_0, \ \overline{m}_0 \in M_\vartheta^\circ(\overline{\eta}_0(\cdot) \mid t, x)$$
$$\varepsilon^\circ(t, x) - \varepsilon^\circ(t_*, x_*) \geqslant \omega(\vartheta, \overline{\varphi}_0(\vartheta), \overline{m}_0) - \omega(\vartheta, \varphi_0(\vartheta), \overline{m}_0)$$
$$\varphi_0(\vartheta) = \varphi(\vartheta, t_*, x_*, \eta_0(\cdot))$$

where $\eta_0(\cdot)$ is any program control of player 1 from $\{\Pi(\nu_0(\cdot)), [t_*, \vartheta]\}$ coinciding with $\overline{\eta}_0(\cdot)$ on $[t, \vartheta] \times P \times Q$.

$$\varphi_0(\vartheta) = \overline{\varphi}_0(\vartheta) + S(\vartheta, t, \overline{\varphi}_0(\cdot), \overline{\eta}_0(\cdot))(\Delta\varphi_0 - \Delta x) + o(\Delta t)$$

where

$$\Delta\varphi_0 = \int_{t_*}^{t}\int_P\int_Q f(t_*, x_*, u, v)\,\bar\eta_0(d\tau \times du \times dv) + o(\Delta t)$$

$$\Delta x = \Big[\int_P\int_Q f(t_*, x_*, u, v)\,\xi(du \times dv)\Big]\Delta t + o(\Delta t)$$

By choosing t from a sufficiently small neighborhood of t_*, we can take it that $\omega(\vartheta, x, m)$ is differentiable in x at the point $(\vartheta, \bar\varphi_0(\vartheta), \bar m_0)$ for every ϑ, $\nu_0(\cdot)$, $\bar\eta_0(\cdot)$ and $\bar m_0$ satisfying inclusions (6.4). Then we can show that

$$\Delta\varepsilon^\circ = \varepsilon^\circ(t, x) - \varepsilon^\circ(t_*, x_*) \geqslant$$

$$\Big[\frac{d}{dx}\omega(\vartheta, \bar\varphi_0(\vartheta), \bar m_0)\Big]' S(\vartheta, t, \bar\varphi_0(\cdot), \bar\eta_0(\cdot))\,(\Delta x - \Delta\varphi_0) - o(\Delta t) +$$

$$\Big[s_*' \int_P\int_Q s(t_*, x_*, u, v)\,\xi(du \times dv)\Big]\Delta t -$$

$$s_*' \int_{t_*}^{t}\int_P\int_Q f(t_*, x_*, u, v)\,\eta_0(d\tau \times du \times dv) + o(\Delta t)$$

for any $s_* \in S^*(t, x \mid t_*, x_*)$ when $\varepsilon^\circ(t, x) \leqslant \varepsilon^\circ(t_*, x_*)$. Since the control $\eta_0(\cdot) \in \{\Pi(\nu_0(\cdot)), [t_*, \vartheta]\}$ can be chosen arbitrarily on $[t_*, t] \times P \times Q$, we obtain the lemma's assertion after simple manipulations with due regard to Condition 3.

7. For each γ by W_γ^* we denote the set of all positions (t, x) for which $\varepsilon^\circ(t, x) \geqslant \gamma$. The set W_γ^* has closed sections $W_\gamma^*(\tau)$. If $\{(t_k, x_k)\} \subset W_\gamma^*$ and $\{t_k\}$ increases monotonically, and, moreover, if $(t_k, x_k) \to (t_*, x_*)$, then $(t_*, x_*) \in W_\gamma^*$. For each probability measure $\mu(\cdot)$ on P, position (t_*, x_*), and instant $t^* \in [t_*, \vartheta_0]$, by $X_\mu(t_*, x_*, t^*)$ we denote the family of program motions on $[t_*, t^*]$ generated by all possible controls $\eta(\cdot) \in \{H(m(\cdot)), [t_*, t^*]\}$ consistent with $\mu(\cdot)$; $\eta(\Gamma \times A \times Q) = m(\Gamma)\mu(A)$ on any Borel sets $\Gamma \subset [t_*, t^*]$ and $A \subset P$.

Lemma 7.1. For every position $(t_*, x_*) \in W_\gamma^*$, $\omega_0 < \gamma < \omega'$, for any probability measure $\mu(\cdot)$ on P, and for any number α, $0 < \alpha < \alpha_0$, there exists a program motion $\varphi_\alpha(t) \in X_\mu(t_*, x_*, t^*)$ on $[t_*, t^*]$ such that $\varepsilon^\circ(t, \varphi_\alpha(t)) \geqslant \gamma - \alpha(t - t_*)$ for any $t \in [t_*, t^*]$.

Lemma 7.2. For every γ, $\omega_0 < \gamma \leqslant \omega^\circ$, the set W_γ^* is v-stable: for any position $(t_*, x_*) \in W_\gamma^*$, probability measure $\mu(\cdot)$ on P, and instant $t^* \in [t_*, \vartheta_0]$ there exists $\varphi^\circ(t) \in X_\mu(t_*, x_*, t^*)$ such that $(t, \varphi^\circ(t)) \in W_\gamma^*$ for $t \in [t_*, t^*]$.

Let the small game [3] possess a saddle point.

Theorem 7.1. If $\omega_0 < \varepsilon^\circ(t_0, x_0) = \varepsilon \leqslant \omega^\circ$, then the strategy V° extremal to W_ε^* solves Problem 3.

Assume that conditions 1 and 2 are fulfilled together with Condition 3 and that U° is a strategy extremal to the system of sets of program absorption $W_\varepsilon(t) = \{x : \varepsilon^\circ(t, x) \leqslant \varepsilon\}$.

Theorem 7.2. If $\omega_0 < \varepsilon^\circ(t_0, x_0) = \varepsilon < \omega^\circ$, then the strategy pair (U°, V°) solves Problem 4; moreover, for any motion $x_{U^\circ, V^\circ}[t]$

$$\min_T \min_{M_\vartheta} \omega(\vartheta, x_{U^\circ, V^\circ}[\vartheta], m) = \varepsilon$$

The author thanks N. N. Krasovskii for posing the problem and for attention to the work.

REFERENCES

1. Batukhtin, V. D., Extremal sighting in a nonlinear game of encounter. Dokl. Akad. Nauk SSSR, Vol. 207, №1, 1972.
2. Krasovskii, N. N., Program absorption in differential games. Dokl. Akad. Nauk SSSR, Vol. 201, №2, 1971.
3. Krasovskii, N. N., A differential encounter-evasion game. Izv. Akad. Nauk SSSR, Tekh. Kibernetika, №2, 1973.
4. Krasovskii, N. N. and Subbotin, A. I., An alternative for the game problem of convergence. PMM Vol. 34, №6, 1970.
5. Tarlinskii, S. I., On a position guidance game. Dokl. Akad. Nauk SSSR, Vol. 207, №1, 1972.
6. Krasovskii, N. N. and Subbotin, A. I., Approximation in a differential game. PMM Vol. 37, №2, 1973.
7. Batukhtin, V. D. and Krasovskii, N. N., Problem of program control by maximin. Izv. Akad. Nauk SSSR, Tekh. Kibernetika, №6, 1972.
8. Ioffe, A. D., Generalized solutions of systems with control. Differentsial'nye Uravneniia, Vol. 5, №6, 1969.

Translated by N. H. C.

UDC 531.31

ON PERIODIC MOTIONS OF A RIGID BODY IN A CENTRAL NEWTONIAN FIELD

PMM Vol. 38, №2, 1974, pp. 224-227

V. G. DEMIN and F. I. KISELEV
(Moscow)

(Received April 21, 1973)

In the problem of the motion of a rigid body with one fixed point in a central Newtonian force field (in particular, in the de Brun field [1]). The existence of a family of periodic solutions is proved by the Poincaré method of small parameters. It is assumed that the body differs negligibly from a dynamically symmetric one and that its center of gravity is sufficiently close to the fixed point. The proof is carried out by using the techniques of Hamiltonian systems.

We investigate the motion of a rigid body around a fixed point in a Newtonian gravity field, making use for this purpose of the canonical Deprit variables [2] which we introduce as follows. Let $OXYZ$ be a fixed coordinate system with origin at a fixed point O, whose Z-coordinate axis is directed vertically upward, and let $Oxyz$ be a system of axes directed along the principal axes of inertia for point O. Further, let φ, ψ, ϑ be the Euler angles defining the position of the moving system $Oxyz$ relative to the fixed one. We introduce a plane containing point O and perpendicular to kinetic moment G. The position of this plane is given by the longitude h of its nodal line on the OXY plane and its inclination I to this same plane. Finally, we introduce two more Euler angles defining the position of the moving system of axes relative to the plane perpendicular to the kinetic moment: the angle of self-rotation l and the nutation angle b.

As coordinates we now take the angles l, g, h introduced. The canonical momenta